APPENDIX A
DERIVING THE JACOBIAN FOR SURFACE NORMALS

This appendix provides a detailed derivation of the Jacobian for nonlinear optimization of surface normals. Since we are trying to optimize the surface normal as a reflectance parameter, we need to find the derivatives of the BRDF with respect to the components of the surface normal $\mathbf{n}$. This problem is complicated by the constraint that $\mathbf{n}$ must be normalized.

We formulate this problem as an optimization of the parameters n_u and n_v, such that $\mathbf{n} = (n_\mathrm{u}, n_\mathrm{v}, n_\mathrm{w}); n_\mathrm{w} = \sqrt{1 - n_\mathrm{u}^2 + n_\mathrm{v}^2}$. We observe that in any direction orthogonal to the current estimate of $\mathbf{n}$, an infinitesimal change to $\mathbf{n}$ in that direction will not affect the length of $\mathbf{n}$. This means that if we define the coordinate axes $\mathbf{u}$ and $\mathbf{v}$ to be orthogonal to the current estimate of $\mathbf{n}$ (i.e., $\mathbf{n} = (0, 0, 1)$ in $u/v/w$ space), the derivatives $\partial n_\mathrm{w}/\partial n_\mathrm{u}$ and $\partial n_\mathrm{w}/\partial n_\mathrm{v}$ are zero. This simplifies the calculation considerably. Every iteration of Levenberg-Marquardt, we redefine the $u/v/w$ coordinate system to maintain this condition. In principle, a Jacobian matrix would need to be introduced into the formulae for the partial derivatives to account for this change of variables; however, since the change of variables is merely an orthogonal linear transformation, the Jacobian is simply the transformation matrix itself. To simplify notation, we will derive the derivatives with respect to the components of the normal vector after the change of variables, with the understanding that these components will be transformed back into the mesh's tangent space after each iteration.

The specular BRDF, according to the work of Nam et al. [1], is a product of an analytic factor (the ratio between the masking / shadowing function and the denominator of the Cook-Torrance model) and a factor for the product of the microfacet distribution D and the Fresnel reflectivity F_0, which is to be approximated using a discrete lookup table with linear interpolation between elements. For the latter factor, we have a function defining the quantity DF_0 as a function of θ_h, the angle between the halfway direction and the surface normal. Given the current value of θ_h (based on the current surface normal), we can first estimate the derivative $\partial(DF_0)/\partial \cos\theta_\mathrm{h}$ as the ratio of discrete differences $\Delta(DF_0)/\Delta\cos\theta_\mathrm{h}$, calculating these differences using the two closest values of θ_h on each side of the current value of θ_h. Then, to convert this quantity into derivatives with respect to the normal vector components, we need to multiply it by the derivatives $\partial\cos\theta_\mathrm{h}/\partial n_\mathrm{u}$ and $\partial\cos\theta_\mathrm{h}/\partial n_\mathrm{v}$. Since $\cos\theta_\mathrm{h}$ is just the inner product of the vectors $\omega_\mathrm{h} = (h_\mathrm{u}, h_\mathrm{v}, h_\mathrm{w})$ and $\mathbf{n}$, we can say:

$$\frac{\partial\cos\theta_\mathrm{h}}{\partial n_\mathrm{u}} = \frac{\partial}{\partial n_\mathrm{u}}(h_\mathrm{u} n_\mathrm{u} + h_\mathrm{v} n_\mathrm{v} + h_\mathrm{w} n_\mathrm{w}) = h_\mathrm{u} \tag{1}$$

By similar reasoning, $\partial\cos\theta_\mathrm{h}/\partial n_\mathrm{v} = h_\mathrm{v}$. Note the importance here of our choice of coordinates, allowing us to assume that $\partial n_\mathrm{w}/\partial n_\mathrm{u}$ and $\partial n_\mathrm{w}/\partial n_\mathrm{v}$ are zero. In summary:

$$\frac{\partial(DF_0)}{\partial n_\mathrm{u}} = h_\mathrm{u}\frac{\Delta(DF_0)}{\Delta\cos\theta_\mathrm{h}} \tag{2}$$

$$\frac{\partial(DF_0)}{\partial n_\mathrm{v}} = h_\mathrm{v}\frac{\Delta(DF_0)}{\Delta\cos\theta_\mathrm{h}} \tag{3}$$

The remaining factors of the specular BRDF, $G/(4\cos\theta_\mathrm{r})$ (note that $\cos\theta_\mathrm{i}$ is canceled out by the fact that the BRDF is cosine-weighted), have derivatives that can be analytically derived and combined with the derivatives of DF_0 via the product rule. Finally, we account for the impact of the surface normal on diffuse reflectance by adding to our derivatives the product of the diffuse albedo and the components of the light direction ω_i in the coordinate directions $\mathbf{u}$ and $\mathbf{v}$, which are the partial derivatives of Lambertian diffuse reflectance with respect to n_u and n_v. The derivatives for the full BRDF are thus as follows:

$$\begin{aligned}\frac{\partial(f\cos\theta_\mathrm{i})}{\partial n_\mathrm{u}} &= f_D\,(\omega_\mathrm{i}\cdot\mathbf{u}) + \frac{G}{4\cos\theta_\mathrm{r}} h_\mathrm{u}\frac{\Delta(DF_0)}{\Delta\cos\theta_\mathrm{h}} \\ &\quad + \frac{DF_0}{4\cos^2\theta_\mathrm{r}}\Big(\cos\theta_\mathrm{r}\frac{\partial G}{\partial n_\mathrm{u}} - G\,(\omega_\mathrm{r}\cdot\mathbf{u})\Big)\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\partial(f\cos\theta_\mathrm{i})}{\partial n_\mathrm{v}} &= f_D\,(\omega_\mathrm{i}\cdot\mathbf{v}) + \frac{G}{4\cos\theta_\mathrm{r}} h_\mathrm{v}\frac{\Delta(DF_0)}{\Delta\cos\theta_\mathrm{h}} \\ &\quad + \frac{DF_0}{4\cos^2\theta_\mathrm{r}}\Big(\cos\theta_\mathrm{r}\frac{\partial G}{\partial n_\mathrm{v}} - G\,(\omega_\mathrm{r}\cdot\mathbf{v})\Big)\end{aligned} \tag{5}$$

If G is calculated using the Smith height-correlated model [2], [3], its derivatives can be defined in terms of the function $\Lambda(\omega, \mathbf{n})$ and its derivative, which are specific to the distribution of microfacets:

$$G = \frac{1}{1 + \Lambda(\omega_\mathrm{r}, \mathbf{n}) + \Lambda(\omega_\mathrm{i}, \mathbf{n})} \tag{6}$$

$$\frac{\partial G}{\partial n_\mathrm{u}} = -\frac{\frac{\partial\Lambda(\omega_\mathrm{r},\mathbf{n})}{\partial n_\mathrm{u}} + \frac{\partial\Lambda(\omega_\mathrm{i},\mathbf{n})}{\partial n_\mathrm{u}}}{(1 + \Lambda(\omega_\mathrm{r}, \mathbf{n}) + \Lambda(\omega_\mathrm{i}, \mathbf{n}))^2} \tag{7}$$

We make the approximation of evaluating masking and shadowing by assuming the parametric GGX model, for which $\Lambda(\omega, \mathbf{n})$ is defined as follows [3]:

$$\Lambda(\omega, \mathbf{n}) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \alpha^2\tan^2\theta} \tag{8}$$

The derivative of $\Lambda(\omega, \mathbf{n})$ can be calculated by letting $\omega = (\omega_\mathrm{u}, \omega_\mathrm{v}, \omega_\mathrm{w})$ and observing that $\cos\theta = \mathbf{n}\cdot\omega$:

$$\Lambda(\omega, \mathbf{n}) = \frac{-1}{2} + \frac{1}{2}\sqrt{1 + \alpha^2\Big(\frac{1}{(n_\mathrm{u}\omega_\mathrm{u} + n_\mathrm{v}\omega_\mathrm{v} + n_\mathrm{w}\omega_\mathrm{w})^2} - 1\Big)} \tag{9}$$

$$\frac{\partial\Lambda(\omega, \mathbf{n})}{\partial n_\mathrm{u}} = \frac{-\alpha^2\,\omega_\mathrm{u}}{2\cos^3\theta\sqrt{1 + \alpha^2\tan^2\theta}} \tag{10}$$

REFERENCES

[1] G. Nam, J. H. Lee, D. Gutierrez, and M. H. Kim, “Practical svbrdf acquisition of 3d objects with unstructured flash photography,” *ACM Transactions on Graphics (TOG)*, vol. 37, no. 6, pp. 267:1–267:12, 2018.

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[3] E. Heitz, “Understanding the masking-shadowing function in microfacet-based brdfs,” *Journal of Computer Graphics Techniques*, vol. 3, no. 2, pp. 32–91, 2014.